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On the Weyl Functional Calculus

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The operator valued distribution $T(A)$ on R^n associated with an n-tuple of self-adjoint operators (A) which was discussed in "The Weyl Functional Calculus" is further studied. The results are:

(1) The support of the distribution $T(A)$ is connected in R^n unless some nontrivial orthogonal projection commutes with all the operators in the n-tuple (A).

(2) In the special case of finite dimensional Hilbert space, $T(A)$ cannot be represented as a measure unless all the operators in the n-tuple (A) commute.

The first result is obtained by studying the propagation of an operator differential equation of which $T(A)$ is a solution.

The second result is obtained by geometric study of the numerical range of an n-tuple of operators.

The results are used to contrast the Weyl calculus for a pair of self-adjoint operators with the closely related Riesz calculus for a bounded operator.

1. INTRODUCTION

Two theorems on the Weyl Functional Calculus will be proved which are not only of intrinsic significance but also help to develop the relationship between the Weyl Calculus and the Riesz calculus for bounded operators on Banach spaces.

The Weyl calculus is discussed in a previous paper [1] of the author. It associates with each n-tuple $A = (A_1, \ldots, A_n)$ of bounded self-adjoint operators in Hilbert space, an operator-valued, compactly supported distribution $T(A)$ on E^n:

$$T(A) \equiv \mathscr{F}^{-1}(\exp(i\xi \cdot A)),$$

where $\mathscr{F}$ denotes Fourier transform, ξ the coordinates of the range space of the transform, and $\xi \cdot A$ the dot product. In particular, if $f(x) = (a \cdot x)^m$, then

$$T(A)f = (a \cdot A)^m.$$

In Section 5 of [1], it is shown that if $n = 2$, and $f(x_1, x_2)$ is analytic in $z = x_1 + ix_2$ on a simply connected domain containing the support of $T(A)$ and the spectrum of $A_1 + iA_2$, then

$$T(A)f = f(A_1 + iA_2)$$

Here $f(A_1 + iA_2)$ is the Riesz function obtained for a contour integral enclosing the spectrum:

$$f(A_1 + iA_2) = \frac{1}{2\pi i} \oint (A_1 + iA_2 - z)^{-1}\, dz.$$

The distribution theoretic approach and the contour integral approach nevertheless have fundamental differences. The natural domain of $T(A)$ is the smooth functions in two real variables, while the contour integrals are well-defined for analytic functions of a complex variable.

However, the Riesz calculus is toplinearly covariant. If C is a bounded operator on a Banach space $\mathscr{B}$, and L is a linear homeomorphism of $\mathscr{B}$ onto $\mathscr{B}$, then $f(L^{-1}CL) = L^{-1}f(C)L$. The Weyl calculus depends on a specific decomposition of C into real and imaginary parts, which is always possible for Hilbert space but depends on the specific inner product used. The Weyl calculus is therefore only a unitary covariant: if U is unitary,

$$f((U^{-1}A_1U, U^{-1}A_2U)) = U^{-1}f((A_1, A_2))\, U.$$

If the spectrum $\sigma(C)$ of a bounded operator C has two components σ_1 and σ_2, there are complementary projections P_1 and P_2 whose ranges are left invariant by C. The Riesz calculus has the important property that $P_j f(C)$ is obtained from the above contour integral by selecting a contour which encloses σ_j only.

Theorem 1 establishes the corresponding facts about the *joint* spectrum, defined as support $T(A)$ and denoted $\sigma_W(A)$. If $\sigma_W(A)$ has two components σ_1 and σ_2, then there are nontrivial complementary orthogonal projections P_1 and P_2 which commute with $A_1, \ldots, A_n$. Also σ_j is the joint spectrum of the n-tuple $(P_jA_1, \ldots, P_jA_n)$.

Theorem 2 shows that the order of the distribution is at least one if $A_1 + iA_2$ is a finite matrix which is not normal. i.e. $A_1 A_2 \neq A_2 A_1$. By contrast, it is obvious from the Jordan canonical form for matrices that

$$\|f(A_1 + iA_2)\| \leqslant \text{const} \sup_{\sigma(A_1+iA_2)} |f|$$

unless $A_1 + iA_2$ has multiple eigenvalues.

Note that the Weyl calculus is formulated for Banach spaces in [1]. and the proofs below are the same for Hilbert and Banach space.

2. Proofs

Definition 1. An n-tuple $A = (A_1, \dots, A_n)$ of bounded self-adjoint operators on a Banach space B is irreducible if there is no non-trivial projection on B which commutes with $A_1, \dots, A_n$.

Theorem 1. *If A is irreducible, its joint spectrum $\sigma_W(A)$ ($\equiv$ support $T(A)$) is connected.*

Proof. Suppose $\sigma_W(A)$ is the disjoint union of closed sets S_1, S_2.

Let $T(A, t) = \mathscr{F}^{-1}[\exp(it\xi \cdot A)]$, t real. Then $T(A, t) = T(tA, 1)$ where $(tA)_j = tA_j$. Since $\sigma_W(tA) = t\sigma_W(A)$ by affine covariance, the support of $T(A, t)$ is the disjoint union of tS_1 and tS_2, $t \neq 0$.

Set the test functions $\varphi_1(x)$, $\varphi_2(x)$ equal 1, 0, respectively, near S_1, and 0, 1, respectively, near S_2. Let $T_j(A, t) = \varphi_j(xt)\, T(A, 1)$. T_j is the restriction of T to S_j.

T propagates itself by convolution as follows:

$$\begin{aligned} T(A, t+s) &= \mathscr{F}^{-1}[\exp(i(t+s)\,\xi \cdot A)] \\ &= \mathscr{F}^{-1}[\exp(it\xi \cdot A)\exp(is\xi \cdot A)] \\ &= \mathscr{F}^{-1}[\mathscr{F}T(A, t)\,\mathscr{F}T(A, s)] \\ &= T(A, t) * T(A, s). \end{aligned} \tag{1}$$

The essential fact to be established is that T propagates T_1 and T_2 separately. First, simply because

$$T = T_1 + T_2, \; t \neq 0$$

$$\begin{aligned} T(A, t+s) &= T_1(A, t) * T(A, s) + T_2(A, t) * T(A, s) \\ &= T(A, s) * T_1(A, t) + T(A, s) * T_2(A, t). \end{aligned}$$

But by definition, when $t+ s \neq 0$,

$$T(A, t + s) = T_1(A, t + s) + T_2(A, t + s).$$

The support of $T_j(A, t) * T(A, s)$ and of $T(A, s) * T_j(A, t)$ is contained in the vector sum $\{s(S_1 \cup S_2) + tS_j\}$ of the supports of $T_j(A, t)$ and $T(A, s)$. The support of $T_k(A, t + s)$ is $\{(t + s) S_k\}$. When $t \neq 0$ and s sufficiently small, these two sets only intersect if $j = k$. Therefore, when $t \neq 0$ and s sufficiently small,

$$\begin{aligned} T_j(A, t + s) &= T_j(A, t) * T(A, s) \\ &= T(A, s) * T_j(A, t). \end{aligned} \tag{2}$$

By iteration of the convolution with small s, it follows that (2) holds whenever t, s have the same sign.

Let the operator M_j be the value of $T_j(A, 1)$ applied to the constant function 1. For any $f \in C^\infty(E^n)$, $\lim_{t\to 0} f(tx) = f(0)\,1$ in the $C^\infty(E^n)$ topology. The equation

$$[T_j(A, t)](f(x)) = [T_j(A, 1)](f(tx))$$

holds for all $f \in C^\infty(E^n)$. Therefore in the limit $t \to 0$ with respect to the weak topology on distributions,

$$\lim_{t\to 0} T_j(A, t) = M_j\delta, \tag{3}$$

where $\delta(f) = f(0)$.

Insertion of (3) into (2) yields

$$T_j(A, t) = M_j T(A, t) \tag{4}$$

$$T_j(A, t) = T(A, t) * M_j\delta. \tag{5}$$

Multiplication of (4) on the left by $\varphi_j(tx)$ yields

$$T_j(A, t) = M_j T_j(A, t)$$

and in the limit $t \to 0$

$$M_j = M_j^2.$$

Thus M_j is a projection.

Finally, applying (4) and (5) to $f(x) = x_i$ with $t = 1$ yields $M_j A_i = A_i M_j$.

Note that by (4),

$$(M_1 + M_2)\, T(A, t) = T_1(A, t) + T_2(A, t) = T(A, t)$$

so that the limit $t \to 0$ yields $M_1 + M_2 = I$.

THEOREM 2. *Let A be an n-tuple of hermitian $m \times m$ matrices. Then $T(A)$ is a distribution of order greater than 0 or the matrices $A_1, \ldots, A_n$ commute.*

Proof. It is shown in [1] that if $T(A)$ is integrated with respect to variables $x_1, \ldots, x_j$, then $T((A_{j+1}, \ldots, A_n))$ is recovered. Since integration does not increase the order of a distribution, it suffices to consider the case $n = 2$. It is also shown in [1] that the numerical range of $A_1 + iA_2$ coincides with the convex hull of $\sigma_W((A_1, A_2))$, if z is identified with (x_1, x_2). The first step is to show that if $T(A)$ is a measure, $\mu(S)$, the convex hull is a polygon.

Suppose $x = c$ is a bounding line for the convex hull. Then c is in the spectrum of A_1. The total variation of μ on $\{(x, y) \mid x > c - \epsilon\}$ is at least 1, since when μ is integrated with respect to y, the spectral measure of A_1 is recovered. By rotation, the same applies to any bounding line of the convex hull. Total variation μ is finite because the convex hull is bounded. Therefore there can only be a finite number of extremal points of the convex hull, which implies it is a polygon.

The map q of the unit vectors in C^2 (a compact manifold) into C^1 given by

$$q(u) = (A_1 u, u) + i(A_2 u, u),$$

is a real analytic map, so whenever $q(u)$ is a vertex of the polygon, the Jacobian of q vanishes at u. Suppose that $q(u_0)$ is the only vertex of the polygon in the right half plane. Then u_0 is an eigenvector, because $(A_1 u_0, u_0)$ is the upper bound for the spectrum of A_1, and a finite matrix has discrete finite spectrum. Since the Jacobian of q has rank 0 at u_0, if $u_1 \perp u_0$ then $d/dt(A_2(u_0 + tu_1), u_0 + tu_1)|_{t=0} = 0$ or $0 = (A_2 u_0, u_1) + (A_2 u_1, u_0) = 2\,\mathrm{Re}(A_2 u_0, u_1)$. Replace u_1 by iu_1 to obtain

$$0 = (A_2 u_0, u_1) - (A_2 u_1, u_0) = 2\,\mathrm{Im}(A_2 u_0, u_1).$$

Therefore $A_2 u_0 = \lambda u_0$.

Because A_1 and A_2 leave the subspace spanned by u_0 invariant, they also leave its orthocomplement invariant. By induction on m the theorem follows.

References

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2. F. Riesz and B. Sz-Nagy, "Functional Analysis," Ungar, New York, 1955.